

SUPERTROPICAL MATRIX ALGEBRA III: POWERS OF MATRICES AND GENERALIZED EIGENSPPACES

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ABSTRACT. We investigate powers of supertropical matrices, with special attention to the role of the coefficients of the supertropical characteristic polynomial (especially the supertropical trace) in controlling the rank of a power of a matrix. This leads to a Jordan-type decomposition of supertropical matrices, together with a generalized eigenspace decomposition of a power of an arbitrary supertropical matrix.

1. INTRODUCTION

This paper develops further the tropical matrix theory from the point of view of supertropical algebras, whose foundation was laid out in [10] and further analyzed in [11]; a treatment from the point of view of linear algebra is to be given in [7].

We recall briefly the main idea behind supertropical matrix theory. Although matrix theory over the max-plus semiring is hampered by the lack of negatives, the use of the “ghost ideal” in supertropical domains enables one to recover most of the classical spirit (and theorems) of matrix theory, and some of the proofs actually become easier than in the classical case. For example, instead of taking the determinant, which requires -1 , one defines the **supertropical determinant** to be the permanent, and we define a matrix to be **nonsingular** when its supertropical determinant is not a ghost. Likewise, vectors are called **tropically dependent** when some linear combination (with tangible coefficients) is a ghost vector; otherwise, they are **tropically independent**.

Prior supertropical results include the fact that the rows (or columns) of a matrix are tropically dependent iff the matrix is nonsingular ([10, Theorem 6.5]), and, more generally, the maximal number of tropically independent rows (or columns) is the same as the maximal size of a nonsingular submatrix, cf. [9]. A key tool is the use of **quasi-identity** matrices, defined below as nonsingular multiplicative idempotent matrices equal to the identity matrix plus a ghost matrix. Quasi-identity matrices were obtained in [11, Theorem 2.8] by means of the adjoint, which was used in [11, Theorems 3.5 and 3.8] to solve equations via a variant of Cramer’s rule, thereby enabling us to compute supertropical eigenvectors in [11, Theorem 5.6]. Furthermore, any matrix A satisfies its **characteristic polynomial** $f_A := |\lambda I + A|$, in the sense that $f_A(A)$ is ghost ([10, Theorem 5.2]), and the tangible roots of f_A turn out to be the supertropical eigenvalues of A ([10, Theorem 7.10]). However, something seems to go wrong, as indicated in [11, Example 5.7], in which it is seen that even when the characteristic polynomial is a product of distinct tangible linear factors, the supertropical eigenvectors need not be tropically independent.

This difficulty can be resolved by passing to asymptotics, i.e., high enough powers of A . In contrast to the classical case, a power of a nonsingular $n \times n$ matrix can be singular (and even ghost). Asymptotics of matrix powers have been studied extensively over the max-plus algebra, as described in [1, Chapter 25.4], but the situation is not quite the same in the supertropical context, since “ghost entries” also play a key role. Whereas [10] and [11] focused on the supertropical determinant, nonsingular matrices and the adjoint, it turns out that the simple cycles contributing to the first coefficient α_μ of the supertropical

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characteristic polynomial, together with other simple cycles of the same average weight, provide the explanation for the phenomena discussed in this paper. This can be described in terms of the **supertropical trace**. Thus, this paper involves a close study of the cycles of the graph of the matrix, one of the main themes of [1, Chapter 25]. Nevertheless, the supertropical point of view leads to somewhat more general results, which were not previously accessible in the language of the max-plus algebra.

There is a reduction to the case where the graph of an $n \times n$ matrix A is strongly connected, in which case the following results hold, cf. Theorems 4.38 and 4.39:

- (a) When all of these leading simple cycles are tangible and disjoint, and their vertex set contains all n vertices, then every power of the matrix A is nonsingular.
- (b) When (a) does not hold and there exists a leading tangible simple cycle, disjoint from all the other leading cycles, then every power of A is non-ghost, but some power is singular.
- (c) When the weight of each of the disjoint leading simple cycles is ghost, then some power of A is ghost.

As indicated above, the ultimate objective of this paper is to show that the pathological behavior described in [11, Example 5.7] can be avoided by passing to high enough powers of the matrix A . In general, there is some change in the behavior of powers up to a certain power A^m , depending on the matrix A , until the theory “stabilizes;” we call this m the **stability index**, which is the analog of the “cyclicity” in [1, Chapter 25]. We see this behavior in the characteristic polynomial, for example, in Theorem 4.2 and Lemma 3.13. The stability index is understood in terms of the leading simple cycles of the graph of A , as explained in Theorem 4.31. A key concept here is the “tangible core” of the digraph of a matrix A , which is the aggregate of those simple cycles which are tangible and disjoint from the others.

Once stability is achieved, the supertropical theory behaves beautifully. Some power of A can be put in full block triangular form, where each block B_i satisfies $B_i^2 = \beta_i B_i$ for some tangible scalar β_i (Corollary 4.33) and the off-diagonal blocks also behave similarly, as described in Theorem 5.7, which might be considered our main result. The special case for a matrix whose digraph is strongly connected is a generalization of [5] to supertropical matrices. (One can specialize to the max-plus algebra and thus rederive their theorem.) These considerations also provide a Jordan-type decomposition for supertropical matrices (Theorem 5.9).

Passing to powers of A leads us to study generalized eigenspaces. A tangible vector v is a **generalized supertropical eigenvector** of A if $A^k v$ equals $\beta^k v$ plus a ghost, i.e. $A^k v = \beta^k v + \text{ghost}$, for some tangible β and some $k \in \mathbb{N}$. (We also include the possibility that $A^k v$ is ghost.) Again, in contrast to the classical theory, the supertropical eigenvalues may change as we pass to higher powers of A , and the theory only becomes manageable when we reach a high enough power A^m of A . In this case, some “thick” subspace of $R^{(n)}$ is a direct sum of generalized eigenspaces of A^m , cf. Theorems 6.7 and 6.8 (although there are counterexamples for A itself). There is a competing concept of “weak” generalized supertropical eigenvectors, using “ghost dependence,” which we consider briefly at the end in order to understand the action of the powers A^k for $k < m$, as indicated in Theorem 6.15. But this transition is rather subtle, and merits further investigation.

2. SUPERTROPICAL STRUCTURES

We recall various notions from [7, 8].

A **semiring without zero**, which we notate as semiring^\dagger , is a structure $(R, +, \cdot, \mathbb{1}_R)$ such that $(R, \cdot, \mathbb{1}_R)$ is a multiplicative monoid, with unit element $\mathbb{1}_R$, and $(R, +)$ is an additive commutative semigroup, satisfying distributivity of multiplication over addition on both sides.

We recall that the underlying supertropical structure is a **semiring[†] with ghosts**, which is a triple (R, \mathcal{G}, ν) , where R is a semiring^\dagger and \mathcal{G} is a semigroup ideal, called the **ghost ideal**, together with an idempotent map

$$\nu : R \rightarrow \mathcal{G}$$

called the **ghost map**, i.e., which preserves multiplication as well as addition and the key property

$$\nu(a) = a + a. \tag{2.1}$$

Thus, $\nu(a) = \nu(a) + \nu(a)$ for all $a \in R$.

We write a^ν for $\nu(a)$, called the ν -**value** of a . Two elements a and b in R are said to be ν -**equivalent**, written $a \cong_\nu b$, if $a^\nu = b^\nu$. (This was called “ ν -matched” in [10].) We write $a \geq_\nu b$, and say that a **dominates** b , if $a^\nu \geq b^\nu$. Likewise we say that a **strictly dominates** b , written $a >_\nu b$, if $a^\nu > b^\nu$.

We define the relation \models_{gs} , called “**ghost surpasses**,” on any semiring with ghosts R , by

$$b \models_{\text{gs}} a \quad \text{iff} \quad b = a \quad \text{or} \quad b = a + \text{ghost}.$$

We write $b \curlyvee_{\text{gd}} a$ if $a + b \in \mathcal{G}$, and say that a and b are **ghost dependent**.

A **supertropical semiring**[†] has the extra properties for all a, b :

- (i) $a + b = a^\nu$ if $a \cong_\nu b$;
- (ii) $a + b \in \{a, b\}$ if $a \not\cong_\nu b$.

A **supertropical domain**[†] is a supertropical semiring[†] for which the **tangible elements** $\mathcal{T} = R \setminus \mathcal{G}$ is a cancellative monoid and the map $\nu_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{G}$ (defined as the restriction from ν to \mathcal{T}) is onto. In other words, every element of \mathcal{G} has the form a^ν for some $a \in \mathcal{T}$.

We also define a **supertropical semifield**[†] to be a supertropical domain[†] (R, \mathcal{G}, ν) in which every tangible element of R is invertible; in other words, \mathcal{T} is a group. In this paper we always assume that R is a supertropical semifield[†] which is **divisible** in the sense that $\sqrt[m]{a} \in R$ for each $a \in R$. With care, one could avoid these assumptions, but there is no need since a supertropical domain[†] can be embedded into a divisible supertropical semifield[†], as explained in [8, Proposition 3.21 and Remark 3.23].

Although in general, the map $\nu : \mathcal{T} \rightarrow \mathcal{G}$ need not be 1:1, we define a function

$$\hat{\nu} : \mathcal{G} \rightarrow \mathcal{T}$$

such that $\nu \circ \hat{\nu} = \text{id}_{\mathcal{G}}$, and write \hat{b} for $\hat{\nu}(b)$. Thus, $(\hat{b})^\nu = b$ for all $b \in \mathcal{G}$. In [11, Proposition 1.6], it is shown that $\hat{\nu}$ can be taken to be multiplicative on the ghost elements, and we assume this implicitly throughout.

It often is convenient to obtain a semiring by formally adjoining a zero element 0_R which is considered to be less than all other elements of R . In this case, R is a semiring with zero element, 0_R , (often identified in the examples with $-\infty$ as indicated below), and the **ghost ideal** $\mathcal{G}_0 = \mathcal{G} \cup \{0_R\}$ is a semiring ideal. We write \mathcal{T}_0 for $\mathcal{T} \cup \{0_R\}$. Adjoining 0_R in this way to a supertropical domain[†] (resp. supertropical semifield[†]) gives us a **supertropical domain** (resp. **supertropical semifield**).

We also need the following variant of the **Frobenius property**:

Proposition 2.1. *Suppose $ab \curlyvee_{\text{gd}} ba$ in a semiring[†] with ghosts. Then $(a + b)^m \models_{\text{gs}} a^m + b^m$ for all m .*

Proof. Any term other than a^m or b^m in the expansion of $(a + b)^m$ has the form $a^{i_1} b^{j_1} \dots$ or $b^{i_1} a^{j_1} \dots$ where $i_1, j_1 \geq 1$. But then we also have the respective terms

$$a^{i_1-1} b a b^{j_1} \dots \quad \text{or} \quad b^{i_1-1} a b a^{j_1} \dots,$$

so summing yields

$$a^{i_1-1} (ab + ba) b^{j_1} \dots \quad \text{or} \quad b^{i_1-1} (ab + ba) a^{j_1} \dots$$

respectively, each of which by hypothesis are in \mathcal{G} . It follows that the sum of all of these terms are in \mathcal{G} . (We do not worry about duplication, in view of Equation (2.1).) \square

Proposition 2.2. *If $q = dm$ for $d > 1$, and a, b commute in a semiring[†] with ghosts, then*

$$(a + b)^q \models_{\text{gs}} (a^m + b^m)^{d-1} \left(\sum_{j=0}^m a^j b^{m-j} \right),$$

with both sides ν -equivalent.

Proof. Both sides are ν -equivalent to $a^q + b^q + (\sum_j a^j b^{q-j})^\nu$, and the left side has more ghost terms. \square

We usually use the algebraic semiring notation (in which $0_R, 1_R$ denote the respective additive and multiplicative identities of R), but for examples occasionally use “logarithmic notation,” in which 1_R is 0 and 0_R is $-\infty$. (Our main example is the extended tropical semiring in which $\mathcal{T} = \mathcal{G} = \mathbb{R}$, cf. [6].)

3. SUPERTROPICAL MATRICES

3.1. Background on matrices. For any semiring[†] R , we write $M_n(R)$ for the set of $n \times n$ matrices with entries in R , endowed with the usual matrix addition and multiplication. The size of the matrices is always denoted n throughout this paper. When R is a semiring[†] with ghosts, we have the ghost map

$$\nu_* : M_n(R) \rightarrow M_n(\mathcal{G}_0)$$

obtained by applying ν to each matrix entry.

When $0_R \in R$, we write (0_R) for the zero matrix of $M_n(R)$. Technically speaking, we need R to have a zero element in order to define the identity matrix and obtain a multiplicative unit in $M_n(R)$, which is itself a semiring. However, in [10] we defined a **quasi-identity** matrix $I_{\mathcal{G}}$ to be a nonsingular multiplicatively idempotent matrix with 1_R on the diagonal and ghosts off the diagonal, and saw that quasi-identity matrices play a more important role in the supertropical theory than identity matrices.

The reader should be aware that the formulations of the results become more complicated when we have to deal with 0_R , which often has to be handled separately. (See, for example, the use of [10, Proposition 6.2] in proving [10, Theorem 6.5].) The use of 0_R leads us to consider strongly connected components in §3.2, and reducible matrices in §3.3.

Let us illustrate the Frobenius property (Proposition 2.1) for matrices.

Example 3.1. Suppose $b = a^2$ and let

$$A = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, \quad B = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Then

$$A^2 = B^2 = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \quad A^2 + B^2 = \begin{pmatrix} b^\nu & 0 \\ 0 & b^\nu \end{pmatrix}, \quad (A + B)^2 = \begin{pmatrix} a & a \\ a & a \end{pmatrix}^2 = \begin{pmatrix} b^\nu & b^\nu \\ b^\nu & b^\nu \end{pmatrix}.$$

We define the supertropical determinant $|A|$ to be the permanent, i.e.,

$$|A| = \sum_{\sigma \in S_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)},$$

as in [6], [9], and [10]. Then

$$|AB| \stackrel{\text{gs}}{=} |A||B|, \tag{3.1}$$

by [10, Theorem 3.5]; a quick proof was found by [2], using their metatheorem which is used to obtain other ghost-surpassing identities, as quoted in [11, Theorem 2.4].

We say that the matrix A is **nonsingular** if $|A|$ is tangible (and thus invertible when R is a supertropical semifield [10]); otherwise, $|A| \in \mathcal{G}_0$ (i.e., $|A| \stackrel{\text{gs}}{=} 0_F$) and we say that A is **singular**. Thus, Equation (3.1) says that $|AB| = |A||B|$ when AB is nonsingular, but there might be a discrepancy when $|AB| \in \mathcal{G}_0$.

One might hope that the “ghost error” in Formula (3.1) might be bounded, say in terms of $|AB|$. But we have the following easy counterexample.

Example 3.2. Let

$$A = \begin{pmatrix} a & 1_R \\ 1_R & 0_R \end{pmatrix}, \quad a >_\nu 1_R. \quad \text{Then } A^2 = \begin{pmatrix} a^2 & a \\ a & 1_R \end{pmatrix},$$

whose determinant is $(a^2)^\nu$ whereas $|A| = 1_R$ (the multiplicative unit). Thus we have no bound for $\frac{|A^2|}{|A|^2} = (a^2)^\nu$, although $|A^2| \stackrel{\text{gs}}{=} |A|^2$.

We also need the following basic fact.

Proposition 3.3. Any multiplicatively idempotent, nonsingular matrix $A = (a_{i,j})$ over a supertropical domain is already a quasi-identity matrix.

Proof. $|A^2| = |A|$ is tangible, implying $|A^2| = |A|^2$, by Equation (3.1), and thus $|A| = \mathbb{1}_R$. The (i, j) entry of A^2 is $a_{i,j} = \sum_k a_{i,k} a_{k,j} = a_{i,i} a_{i,j} + a_{i,j} a_{j,j} + \sum_{k \neq i,j} a_{i,k} a_{k,j}$. Thus, for $i = j$ we have $a_{i,i} \leq_\nu a_{i,i}^2$.

On the other hand, by impotency of A , $a_{i,i} a_{i,j} \leq_\nu a_{i,j}$, implying each $a_{i,i}$ is tangible (since otherwise $a_{i,j} \in \mathcal{G}_0$ for each j , implying A is singular, contrary to hypothesis).

Also, taking $i = j$ yields $a_{i,i}^2 \leq_\nu a_{i,i}$, implying $a_{i,i} \cong_\nu a_{i,i}^2$, and since $a_{i,i} \neq \mathbb{0}_R$ is tangible, we must have each $a_{i,i} = \mathbb{1}_R$. But then for $i \neq j$ we now have $a_{i,j} = a_{i,j}^\nu + \sum_{k \neq i,j} a_{i,k} a_{k,j}$, and thus $a_{i,j}$ is a ghost. \square

Remark 3.4. *It easy to verify that changing one (or more) of the diagonal entries of a quasi-identity matrix I_G to be $\mathbb{1}_R^\nu$, we get a singular idempotent matrix J_G with $J_G \models_{\text{gs}} I_G$.*

3.2. The weighted digraph. As described in [1], one major computational tool in tropical matrix theory is the **weighted digraph** $G_A = (\mathcal{V}, \mathcal{E})$ of an $n \times n$ matrix $A = (a_{i,j})$, which is defined to have vertex set $\mathcal{V} = \{1, \dots, n\}$ and an edge (i, j) from i to j (of **weight** $a_{i,j}$) whenever $a_{i,j} \neq \mathbb{0}_R$. We write $\#(\mathcal{V})$ for the number of elements in the vertex set \mathcal{V} .

As usual, a **path** p (called “walk” in [1]) of **length** $\ell = \ell(p)$ in a graph is a sequence of ℓ edges $(i_1, i_2), (i_2, i_3), \dots, (i_\ell, i_{\ell+1})$, which can also be viewed as a sequence of the vertices $(i_1, \dots, i_{\ell+1})$. For example, the path $(1, 2, 4)$ starts at vertex 1, and then proceeds to 2, and finally 4. A **cycle** is a path with the same initial and terminal vertex. Thus, $(1, 2, 4, 3, 2, 5, 1)$ is a cycle.

We say that vertices $i, j \in \mathcal{V}(G_A)$ are **connected** if there is a path from i to j ; the vertices i and j are **strongly connected** if there is a cycle containing both i and j ; in other words, there is a path from i to j and a path from j to i . The **strongly connected component** of a vertex i is the set of vertices strongly connected to i .

The matrices A that are easiest to deal with are those for which the entire graph G_A is strongly connected, i.e., any two vertices are contained in a cycle. Such matrices are called **irreducible**, cf. [1].

Reducible matrices are an “exceptional” case which we could avoid when taking matrices over supertropical domains[†], i.e., domain without zero. Nevertheless, in order to present our results as completely as we can, we assume from now on that we are taking matrices over a supertropical domain R (with $\mathbb{0}_R$).

Compressing each strongly connected component to a vertex, one obtains the induced **component digraph** \tilde{G}_A of G_A , and thus of A , which is an acyclic digraph. The number of vertices of \tilde{G}_A equals the number of strongly connected components of G_A . Note that the graph \tilde{G}_A is connected iff G_A is connected.

A **simple cycle**, written as **scycle**, is a cycle having in-degree and out-degree 1 in each of its vertices [10, §3.2]. For example, the cycle $(1, 3, 1)$ is simple, i.e., whereas the cycle $(1, 3, 5, 3, 1)$ is not simple.

A **k -multicycle** of G_A is a disjoint union of scycles the sum of whose lengths is k . The **weight** of a path p , written $w(p)$, is the product of the weights of its edges (where we use the semiring operations); the **average weight** of a path p is ${}^{\ell(p)}\sqrt{w(p)}$. A path p is called **tangible** if $w(p) \in \mathcal{T}$; otherwise, p is called **ghost**.

Given a subgraph G' of G_A , we write $\mathcal{V}(G')$ for the set of vertices of G' . Given a scycle C passing through vertices i, j , we write $C(i, j)$ for the subpath of C from i to j . Thus, the cycle C itself can be viewed as the subgraph $C(i, i)$ for any vertex $i \in \mathcal{V}(C)$.

By **deleting** a scycle $C = C(i, i)$ from a path p , we mean replacing C by the vertex i . For example, deleting the cycle $(3, 6, 3)$ from the path $(1, 2, 3, 6, 3, 5)$ yields the path $(1, 2, 3, 5)$. Similarly, **inserting** a scycle $C = C(i, i)$ into p means replacing the vertex i by the cycle C .

3.3. Block triangular form. The **submatrix** (of A) corresponding to a subset $\{i_1, \dots, i_k\}$ of vertices of the graph G_A is defined to be the $k \times k$ submatrix of A obtained by taking the i_1, \dots, i_k rows and columns of A .

We say A has **full block triangular form** if it is written as

$$A = \begin{pmatrix} B_1 & B_{1,2} & \cdots & B_{1,\eta-1} & B_{1,\eta} \\ (0) & B_2 & \cdots & B_{2,\eta-1} & B_{2,\eta} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (0) & \cdots & (0) & B_{\eta-1} & B_{\eta-1,\eta} \\ (0) & \cdots & (0) & (0) & B_\eta \end{pmatrix}, \quad (3.2)$$

where each diagonal block B_i is an irreducible $n_i \times n_i$ matrix, $i = 1, \dots, \eta$, and each $B_{i,j}$, $j > i$, is an $n_i \times n_j$ matrix. (Here we write (0) for the submatrices (0_R) in the appropriate positions.) Thus, in this case, the component graph \tilde{G}_A of A is acyclic and has η vertices.

Proposition 3.5. *A matrix A is reducible iff it can be put into the following form (renumbering the indices if necessary):*

$$A = \begin{pmatrix} B_1 & C \\ (0) & B_2 \end{pmatrix}, \quad (3.3)$$

where $n = k + \ell$, B_1 is a $k \times k$ matrix, C is a $k \times \ell$ matrix, B_2 is an $\ell \times \ell$ matrix, and (0) denotes the zero $\ell \times k$ matrix.

More generally, any matrix A can be put into full block triangular form as in (3.2) (renumbering the indices if necessary), where the diagonal blocks B_i correspond to the strongly connected components of A . In this case,

$$|A| = |B_1| \cdots |B_\eta|.$$

In particular, A is nonsingular iff each B_i is nonsingular.

Proof. Obviously any matrix in the form of (3.3) is reducible. Conversely, suppose that A is reducible. Take indices i, j with no path from j to i . Let $\mathcal{I} \subset \{1, \dots, n\}$ denote the set of indices of G_A having a path terminating at i , and $\mathcal{J} = \{1, \dots, n\} \setminus \mathcal{I}$. Renumbering indices, we may assume that $\mathcal{I} = \{1, \dots, \ell\}$ and $\mathcal{J} = \{\ell + 1, \dots, n\}$ for some $1 \leq \ell < n$. A is in the form of (3.3) with respect to this renumbering, and iterating this procedure puts A in full block triangular form. \square

Remark 3.6. *If A is in full block triangular form as in (3.2), then*

$$A^m = \begin{pmatrix} B_1^m & ?? & ?? & \cdots & ?? \\ (0) & B_2^m & ?? & \cdots & ?? \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (0) & \cdots & (0) & B_{\eta-1}^m & ?? \\ (0) & \cdots & (0) & (0) & B_\eta^m \end{pmatrix}. \quad (3.4)$$

It follows that $|A^m| = |B_1^m| \cdots |B_\eta^m|$, for any $m \in \mathbb{N}$.

3.4. Polynomials evaluated on matrices. Recall that in the supertropical theory we view polynomials as functions, and polynomials are identified when they define the same function. Suppose a polynomial $f = \sum_i \alpha_i \lambda^i$ is a sum of monomials $\alpha_i \lambda^i$. Let $g = \sum_{i \neq j} \alpha_i \lambda^i$. The monomial $\alpha_j \lambda^j$ is **inessential** in f , iff $f(a) = g(a)$ for every $a \in R$. An inessential monomial h of f is **quasi-essential** if $f(a) \cong_\nu h(a)$ for some point $a \in R$. The **essential part** f^{es} of a polynomial $f = \sum \alpha_i \lambda^i$ is the sum of those monomials $\alpha_j \lambda^j$ that are essential.

A polynomial $f \in R[\lambda]$ is called **primary** if it has a unique corner root (cf. [8, Lemma 5.10]), up to ν -equivalence.

Lemma 3.7. *If $f \in R[\lambda]$ is primary, then $f \cong_\nu \alpha \sum a^i \lambda^{d-i}$, where $\alpha \lambda^d$ is the leading monomial of f . If moreover $f \in R[\lambda]$ is monic primary with constant term a^d , then $(\lambda + a)^d \underset{\text{gs}}{=} f$.*

Proof. The first assertion is obtained by writing f as a sum of quasi-essential monomials and observing that corner roots are obtained by comparing adjacent monomials of f .

The second assertion follows by expanding $(\lambda + a)^d = \lambda^d + a^d + \sum_{i=1}^{d-1} (a^i)^\nu \lambda^{d-i}$. \square

We say that a matrix A **satisfies** a polynomial $f \in R[\lambda]$ if $f(A) \models_{\text{gs}} (0)$; i.e., $f(A)$ is a ghost matrix.

In particular, A satisfies its **characteristic polynomial**

$$f_A := |\lambda I + A| = \lambda^n + \sum \alpha_k \lambda^{n-k},$$

where α_k is the sum of all k -multicycles in the graph G_A of the matrix A , cf. [10, Theorem 5.2]; those multicycles of largest ν -value are called the **dominant k -multicycles** of the characteristic coefficients of f_A .

The **essential characteristic polynomial** is defined to be the essential part f_A^{es} of the characteristic polynomial f_A , cf. [8, Definition 4.9]. The **tangible characteristic polynomial** $\widehat{f_A}$ of A is defined as

$$\widehat{f_A} := \sum_{k=0}^n \widehat{\alpha_k} \lambda^k.$$

Writing $f_A = \lambda^n + \sum_{k=1}^n \alpha_k \lambda^{n-k}$, we take

$$L(A) := \{\ell \geq 1 : \sqrt[\ell]{\alpha_\ell} \geq \sqrt[k]{\alpha_k} \text{ for each } k \leq n\}. \quad (3.5)$$

(There may be several such indices.) In other words, $\ell \in L$ if some ℓ -multicycle of A has dominating average weight, either tangible or ghost weight.

Definition 3.8. We define

$$\mu(A) := \min\{\ell \mid \ell \in L(A)\}, \quad (3.6)$$

and call α_μ the **leading characteristic coefficient** of A , which we say is of **degree** $\mu(A)$. We denote $\mu(A)$ as μ if A is understood. We define the **leading (tangible) average weight** $\omega := \omega(A)$ to be

$$\omega(A) := \sqrt[\mu]{\alpha_\mu}.$$

When A is tangible, $\sqrt[\mu]{\alpha_\mu}$ is the “maximal cycle mean” $\rho_{\max}(A)$ in the sense of [1].

Remark 3.9. If A is in full block triangular form as in (3.2), then, by Remark 3.6 the characteristic polynomial of A^m is the product of the characteristic polynomials of the B_i^m , so many properties of A^m can be obtained from those of the B_i^m .

We define the **(supertropical) trace** $\text{tr}(A)$ of the matrix $A = (a_{i,j})$ to be

$$\text{tr}(A) := \sum_{i=1}^n a_{i,i}.$$

Remark 3.10. Note that $\alpha_1 = \text{tr}(A)$. If $\alpha_1 \lambda^{n-1}$ is an essential or a quasi-essential monomial of f_A , then $\text{tr}(A)$ is the leading characteristic coefficient of A , and $\mu = 1$. Furthermore, taking $\beta \in \mathcal{T}$ ν -equivalent to α_1 , we know that β dominates all the other supertropical eigenvalues of A , and thus in view of [10, Theorem 7.10] is the eigenvalue of highest weight.

Example 3.11. The characteristic polynomial of any $n \times n$ quasi-identity matrix I_G is

$$\lambda^n + \sum_{i=1}^{n-1} \mathbb{1}_R^\nu \lambda^{n-i} + \mathbb{1}_R,$$

since any scycle contributing a larger characteristic coefficient could be completed with $\mathbb{1}_R$ along the diagonal to a dominating ghost contribution to $|I_G|$, contrary to the fact that $|I_G| = \mathbb{1}_R$. (This argument is implicit in [10, Remark 4.2].) Hence, $\mu = 1$, and the leading characteristic coefficient of I_G is $\text{tr}(I_G)$, which is $\mathbb{1}_R^\nu$.

Lemma 3.12. If $f = \sum \alpha_i \lambda^i$, then $f(A)^m \models_{\text{gs}} g(A^m)$, where $g = \sum \alpha_i^m \lambda^i$.

Proof. By Proposition 2.1,

$$f(A)^m = \left(\sum \alpha_i A^i \right)^m \models_{\text{gs}} \sum \alpha_i^m A^{im} = g(A^m),$$

again by Proposition 2.1. □

Lemma 3.13. *If A is as in Remark 3.6, then $f_{A^m} = f_{B_1^m} \cdots f_{B_\eta^m}$.*

Proof. $f_{A^m} = |\lambda I + A^m|$, which is the product of the determinants of the diagonal blocks of $\lambda I + A^m$, i.e., the $|\lambda I + B_j^m|$. \square

4. POWERS OF MATRICES

We would like to study powers of a matrix $A \in M_n(R)$, where R is a supertropical domain, in terms of properties of its characteristic polynomial f_A . Certain properties can be had quite easily.

Lemma 4.1. *If A satisfies the polynomial $f = \sum \alpha_i \lambda^i$, then A^m satisfies the polynomial $\sum \alpha_i^m \lambda^i$.*

Proof. This is a special case of Lemma 3.12. \square

Theorem 4.2. *If the characteristic polynomial $f_A = \sum_{i=0}^n \alpha_i \lambda^i$, then $f_{A^m} \models_{\text{gs}} \sum_{i=0}^n \alpha_i^m \lambda^i$, for any m .*

Proof. Application of Lemma 4.1 to [10, Theorem 5.2]. \square

Idempotent matrices need not have diagonal entries ν -equivalent to 1_R ; for example the matrix

$$A = \begin{pmatrix} -1 & -2 \\ 1 & 0 \end{pmatrix}$$

(in logarithmic notation) is idempotent, but also singular; i.e., $|A| = (-1)^\nu$. Later, cf. Lemma 4.29, we see that this is impossible for nonsingular matrices.

Example 4.3. *Suppose*

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$$

in logarithmic notation. Then $\text{tr}(A) = 2$ and $|A| = 2$, so

$$f_A = \lambda^2 + 2\lambda + 2 = (\lambda + 2)(\lambda + 0).$$

Note that $\mu(A) = 1$ and $\alpha_1 = 2$.

On the other hand,

$$A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix};$$

$\text{tr}(A^2) = 4$ and $|A|^2 = 5^\nu$, so

$$f_{A^2} = \lambda^2 + 4\lambda + 5^\nu.$$

By Lemma 4.1, A^2 also satisfies the polynomial $\lambda^2 + 4\lambda + 4$. For A^4 we then have

$$A^4 = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = 4A^2,$$

and in general $A^{2k} = (A^4)^k = 4^k A^2 = 2^{2k} A^2$.

4.1. Ghostpotent matrices. In Example 3.2, no power of A is ghost, and we would like to explore such a phenomenon.

Definition 4.4. *The matrix A is **ghostpotent** if $A^m \in M_n(\mathcal{G}_0)$, i.e., $A^m \models_{\text{gs}} (0)$, for some $m > 0$. The least such m is called the **ghost index** of the ghostpotent matrix A .*

It is easy to check that if A is ghost (i.e., has ghost index 1) so is AB for any $B \in M_n(R)$, and therefore if A is ghostpotent with ghost index m then A^k is ghost for any $k \geq m$. However, in contrast to the classical theory, the ghost index of an $n \times n$ ghostpotent matrix need not be $\leq n$. Although in cf. [7, Theorem 3.4] it is shown that the product of two nonsingular matrices cannot be ghost, a ghostpotent matrix can still be nonsingular.

Example 4.5. The nonsingular matrix $A = \begin{pmatrix} \mathbb{0}_R & \mathbb{1}_R \\ \mathbb{1}_R & \mathbb{1}_R \end{pmatrix}$ is ghostpotent, for which $A^2 = \begin{pmatrix} \mathbb{1}_R & \mathbb{1}_R \\ \mathbb{1}_R & \mathbb{1}_R \end{pmatrix}$ is singular, $A^3 = \begin{pmatrix} \mathbb{1}_R & \mathbb{1}_R^\nu \\ \mathbb{1}_R^\nu & \mathbb{1}_R \end{pmatrix}$, and only for $m = 4$ do we obtain the ghost matrix $A^4 = \begin{pmatrix} \mathbb{1}_R^\nu & \mathbb{1}_R^\nu \\ \mathbb{1}_R^\nu & \mathbb{1}_R^\nu \end{pmatrix}$ (which is ν -equivalent to A^2). Note here that $f_A = \lambda^2 + \lambda + \mathbb{1}_R$, and $\mu(A) = 1$, even though the monomial λ is not essential, but only quasi-essential.

From this example, we see that the image of the action of a ghostpotent matrix A on a vector space can be a thick subspace [7, §5.5], for A can be nonsingular.

Example 4.6. Let $A = \begin{pmatrix} \mathbb{1}_R & a \\ b & \mathbb{1}_R \end{pmatrix}$.

- (i) When $ab <_\nu \mathbb{1}_R$, the matrix $A^2 = \begin{pmatrix} \mathbb{1}_R & a^\nu \\ b^\nu & \mathbb{1}_R \end{pmatrix}$ is nonsingular idempotent (in fact a quasi-identity matrix), and thus not ghostpotent.
- (ii) When $ab \cong_\nu \mathbb{1}_R$, then $A^2 = \begin{pmatrix} \mathbb{1}_R^\nu & a^\nu \\ b^\nu & \mathbb{1}_R^\nu \end{pmatrix}$, which is already ghost.
- (iii) When $ab >_\nu \mathbb{1}_R$, then $A^2 = \begin{pmatrix} ab & a^\nu \\ b^\nu & ab \end{pmatrix}$ and $A^4 = abA^2$. Thus, in this case, A is ghostpotent iff a or b is a ghost.

Here is an instance where one does have a strong bound on the ghost index.

Proposition 4.7. If the only supertropical eigenvalue of A is $\mathbb{0}_R$, then $A^n \models_{\text{gs}} (\mathbb{0}_R)$.

Proof. The characteristic polynomial f_A of A cannot have any roots other than $\mathbb{0}_R$, and thus $f_A = \lambda^n$. Hence, $A^n \in M_n(\mathcal{G}_0)$ by [10, Theorem 5.2]. \square

The following result enables us to reduce ghostpotence to irreducible matrices. (We write (0) for $(\mathbb{0}_R)$.)

Lemma 4.8. If

$$N = \begin{pmatrix} N_1 & ? & ? & \dots & ? \\ (0) & N_2 & ? & \dots & ? \\ \vdots & \vdots & \ddots & \dots & \vdots \\ (0) & \dots & (0) & N_{\eta-1} & ? \\ (0) & \dots & (0) & (0) & N_\eta \end{pmatrix},$$

where $N_i^{m_i}$ is ghost, then $N^{\eta m} \in M_n(\mathcal{G}_0)$ for any $m \geq \max\{m_1, \dots, m_\eta\}$.

Proof.

$$N^m = \begin{pmatrix} N_1^m & ?? & ?? & \dots & ?? \\ (0) & N_2^m & ?? & \dots & ?? \\ \vdots & \vdots & \ddots & \dots & \vdots \\ (0) & \dots & (0) & N_{\eta-1}^m & ?? \\ (0) & \dots & (0) & (0) & N_\eta^m \end{pmatrix},$$

which is ghost on the diagonal blocks, and the η power of this matrix makes everything ghost. \square

Theorem 4.39 will give us a complete determination of ghostpotent matrices.

4.2. Computing powers of matrices. Our next concern is to compute powers of a matrix A , in order to determine whether some power A^m of A is a singular matrix (or even ghost), and, if so, to determine the minimal such m . There is no bound on the power we might need to get a singular matrix.

Example 4.9.

Let $A = \begin{pmatrix} a & \mathbb{1}_R \\ \mathbb{1}_R & b \end{pmatrix}$, where $a >_\nu b >_\nu \mathbb{1}_R$ are tangible, and thus A is nonsingular. Then

$$A^k = \begin{pmatrix} a^k & a^{k-1} \\ a^{k-1} & a^{k-2} + b^k \end{pmatrix} = a^{k-1} \begin{pmatrix} a & \mathbb{1}_R \\ \mathbb{1}_R & a^{-1} + \frac{b^k}{a^{k-1}} \end{pmatrix}.$$

We are interested in the lower right-hand term. This is $\frac{b^k}{a^{k-1}}$ (and thus tangible) so long as it dominates a^{-1} . On the other hand, if $b^k \cong_\nu a^{k-2}$ then

$$A^k = a^{k-1} \begin{pmatrix} a & \mathbb{1}_R \\ \mathbb{1}_R & (a^{-1})^\nu \end{pmatrix},$$

and if $b^k <_\nu a^{k-2}$ then

$$A^k = a^{k-1} \begin{pmatrix} a & \mathbb{1}_R \\ \mathbb{1}_R & a^{-1} \end{pmatrix},$$

both of which are singular.

In other words, taking tangible $c >_\nu \mathbb{1}_R$, if $a = c^k$ and $b = c^{k-2}$, then A^{k-1} is nonsingular whereas A^k is singular.

Nevertheless, one can get information from the leading characteristic coefficient. We write μ for $\mu(A)$, cf. Definition 3.8. As noted following [10, Definition 5.1], the coefficient α_ℓ of the characteristic polynomial f_A is the sum of the weights of the ℓ -multicycles in the graph G_A . Thus, the leading characteristic coefficient α_μ of A is the sum of the weights of the multicycles of length μ (also having maximal average weight, whose tangible value is denoted as ω) in the weighted digraph G_A of A . Accordingly, let us explore the multicycles contributing to α_μ .

Lemma 4.10. *Any multicycle contributing to the leading characteristic coefficient must be a scycle.*

Proof. If some dominant such multicycle were not a scycle, it could be subdivided into smaller disjoint scycles, at least one of which would have average weight $\geq_\nu \omega$ (and a shorter length) and thus which would give a leading characteristic coefficient of lower degree, contrary to the definition of leading characteristic coefficient. \square

Thus, we can focus on scycles.

Definition 4.11. A **leading scycle** is a scycle whose average weight is ν -equivalent to ω . A **leading ℓ -scycle** is a leading scycle of length ℓ . (In particular, α_ℓ equals the sum of the weights of the leading ℓ_i -scycles with $\sum \ell_i = \ell$.) The number of leading ℓ -scycles is denoted τ_ℓ . Given an index i , we define its **depth** ρ_i to be the number of leading scycles of G_A containing i .

In view of (3.6), the length ℓ of a leading ℓ -scycle must be between μ and the least degree of monomials in the essential characteristic polynomial. In Example 4.5 there are leading scycles of length both 1 and 2. Note that $\rho_i = \mu = 1$ where $\tau_i = 1$, for $i = 1, 2$.

Example 4.12. Any quasi-identity matrix satisfies $\mu = \rho_i = 1$ for each i , whereas $\tau_\mu = n$.

Lemma 4.13. If the leading characteristic coefficient α_μ of A is tangible, then $\tau_\mu = 1$.

Proof. Otherwise, α_μ would be the sum of several ν -equivalent weights, and thus must be ghost. \square

Lemma 4.14. $\mu(A^\mu) = 1$, for any matrix A .

Proof. Let $A^\mu = (b_{i,j})$, and let $C_\mu = C(i, i)$ be a leading μ -scycle of A . Then, $b_{i,i} = w(C_\mu)$, and (i, i) is a 1-multicycle of A^μ , which is comprised of just one scycle of length 1, and is clearly a leading scycle of A^μ . \square

Definition 4.15. A scycle is **core-admissible** if each of its vertices has depth 1; i.e., it is disjoint from each other leading scycle. The **core** of an irreducible matrix A , written $\text{core}(A)$, is the multicycle comprised of the union of all core-admissible leading scycles. The **tangible core** of A , written $\text{tcore}(A)$, is the multicycle comprised of the union of all tangible core-admissible leading scycles.

Thus, a leading scycle is part of the core iff its vertex set is disjoint from all other leading scycles in A . Note that $\text{tcore}(A) \subseteq \text{core}(A)$, and also note that $\text{core}(A)$ and $\text{tcore}(A)$ can be empty; for example the core of $A = \begin{pmatrix} \mathbb{1}_R & \mathbb{1}_R \\ \mathbb{1}_R & \mathbb{1}_R \end{pmatrix}$ is empty.

We write $(A)_{\text{core}}$ (resp. $(A)_{\text{tcore}}$) to denote the submatrix of A comprised of the rows and columns corresponding to the indices of $\mathcal{V}(\text{core}(A))$ (resp. $\mathcal{V}(\text{tcore}(A))$).

The idea behind the core is that the vertex set of $\text{tcore}(A)$ is comprised precisely of those vertices of leading scycles which contribute tangible weights to high powers of A . The other vertices of leading scycles provide the “ghost part,” so we also consider them.

Definition 4.16. The *anti-tangible-core*, denoted $\text{anti-tcore}(A)$, is the multicyle comprised of the union of those leading scycles which are not in $\text{tcore}(A)$. We write $(A)_{\text{anti-tcore}}$ to denote the submatrix of A comprised of the rows and columns corresponding to $\mathcal{V}(\text{anti-tcore}(A))$.

The anti-tcore is the set of vertices i for which the (i, i) entry of high powers of A become ghosts. (Note that $\text{core}(A) \cap \text{anti-tcore}(A)$ could be nonempty, when A has a core-admissible cycle with ghost weight.)

The leading scycles appear in the following basic computation.

Remark 4.17. The (i, j) entry of a given power A^m of A is the sum of weights of all paths of length m from i to j in G_A . By the pigeonhole principle, any path p of length $\geq n$ contains an scycle C (since some vertex must repeat, so $w(p) = w(C)w(p')$ where p' is the path obtained by deleting the scycle C from p . Continuing in this way enables us to write $w(p)$ as the product of weights of scycles times a simple path of length $< n$ from i to j .

If $i = j$, then $w(p)$ can thereby be written as a product of weights of scycles. Since, by definition, the average weight of each scycle must have ν -value at most ω , the weight of p is at most w^m , the maximum being attained when all the scycles are leading scycles.

If $i \neq j$, one has to consider all paths of length $< n$ from i to j and take the maximum weight in conjunction with those of the scycles; this is more complicated, but we only will need certain instances.

Example 4.18. Taking

$$A = \begin{pmatrix} 0_R & 0_R & 1_R & 1_R & 0_R \\ 0_R & 0_R & 0_R & 0_R & 1_R \\ 0_R & 1_R & 0_R & 0_R & 0_R \\ 0_R & 1_R & 0_R & 0_R & 0_R \\ 1_R & 0_R & 0_R & 0_R & 0_R \end{pmatrix},$$

we have $\mathcal{V}(\text{core}(A)) = \emptyset$ since the two leading scycles $(1, 3, 2, 5)$ and $(1, 4, 2, 5)$ intersect at the vertex 2. But

$$A^2 = \begin{pmatrix} 0_R & 1_R^\nu & 0_R & 0_R & 0_R \\ 1_R & 0_R & 0_R & 0_R & 0_R \\ 0_R & 0_R & 0_R & 0_R & 1_R \\ 0_R & 0_R & 0_R & 0_R & 1_R \\ 0_R & 0_R & 1_R & 1_R & 0_R \end{pmatrix},$$

and so $\mathcal{V}(\text{core}(A^2)) = \{1, 2\}$ whereas still $\mathcal{V}(\text{tcore}(A^2)) = \emptyset$ since the leading two tangible scycles $(3, 5)$ and $(4, 5)$ intersect.

Thus, our next result is sharp. By C^k we mean the concatenation of C taken k times; for example,

$$(1, 3, 5, 2, 1)^3 = (1, 3, 5, 2, 1, 3, 5, 2, 1, 3, 5, 2, 1).$$

In the reverse direction, we can decompose a cycle into a union of other cycles by “skipping” vertices. For example, skipping two vertices each times decomposes $(1, 3, 5, 2, 1, 4, 5, 2, 1, 3, 4, 2, 1)$ into the three cycles $(1, 2, 5, 3, 1)$, $(3, 1, 2, 4, 3)$, and $(5, 4, 1, 2, 5)$.

Proposition 4.19.

- (i) If C is a leading scycle in G_A having average weight ω , then for any $k \in \mathbb{N}$, the cycle C^k decomposes into a union of leading scycles for G_{A^k} each having average weight ω^k , where we take every k vertex, as indicated in the proof.
- (ii) $\mathcal{V}(\text{core}(A)) \subseteq \mathcal{V}(\text{core}(A^k))$ and $\mathcal{V}(\text{tcore}(A)) \subseteq \mathcal{V}(\text{tcore}(A^k))$, for any $k \in \mathbb{N}$.
- (iii) $\mathcal{V}(\text{tcore}(A)) = \mathcal{V}(\text{tcore}(A^k))$, for any $k \in \mathbb{N}$.

Proof. (i) Let $C = (i_1, \dots, i_\mu)$. Then $C^k = (i_1, \dots, i_\mu) \cdots (i_1, \dots, i_\mu)$, which in G_{A^k} appears as

$$(i_1, i_{1+k}, \dots)(i_{j_2}, i_{j_2+k}, \dots) \cdots \quad (4.1)$$

But $w(C') \leq_\nu \omega^k$ in G_{A^k} , since otherwise we could extract a cycle in G_A of weight $>_\nu \omega$, contrary to hypothesis. It follows that each cycle in (4.1) has average weight ω^k .

(ii) If an index $i \in \mathcal{V}(\text{core}(A))$ appears in a single leading scycle C of G_A then it appears in the corresponding leading scycle of G_{A^k} , according to (i), and cannot appear in another one, since otherwise we could extract another leading scycle of G_A whose vertex set intersects that of C , contrary to hypothesis. Furthermore, if C has tangible weight, then so does C^k .

(iii) The same argument as in Remark 4.17 shows that, for any m and any leading scycle C' of A^k , we can extract leading scycles of A until we obtain the empty set. If their vertices are all in $\mathcal{V}(\text{core}(A))$, then clearly $\mathcal{V}(C') \subseteq \mathcal{V}(\text{core}(A^k))$, as desired. \square

Lemma 4.20. *There is a number $m = m(n, \mu)$ such that, for any path p of length $\geq m$ from vertices i to j in G_A having maximal weight and for which its vertex set $\mathcal{V}(p)$ intersects $\mathcal{V}(\text{core}(A))$, p contains a set of leading scycles of total length a multiple $\ell\mu$ for some $\ell \leq n$.*

When $\mu = 1$, we can take $m = 2n - 1$. In general, we can take $m = \binom{n+1}{2} - \mu(\mu - 1) + 2(n - 1) + 1$.

Proof. Take a leading scycle C_μ of length μ , with $k \in \mathcal{V}(C_\mu) \cap \mathcal{V}(p)$. We are done (with $\ell = 1$) if p contains a scycle C of length μ , since we could delete C and insert C_μ at vertex k , contrary to the hypothesis of maximal weight unless C itself is a leading scycle. Thus, we may assume that p contains no scycle of length μ . Next, any path of length $\geq n$ starting from k contains a scycle that could be extracted, so if the path has length $\geq s + (n - 1)$ we could extract scycles of total length at least s ; likewise for any path ending at k .

Thus, for $\mu = 1$, we could take $s = 1$ and $m = (n - 1) + (n - 1) + 1 = 2n - 1$. Then we are able to extract a scycle of some length $\ell \leq n$, which we could replace by ℓ copies of C_μ each of which is a leading scycle.

In general, if p has length at least $\binom{n+1}{2} - \mu(\mu - 1) + 2(n - 1) + 1$, using the pigeonhole principle, we are able to extract μ scycles of some length $\ell \leq n$, which we could replace by ℓ copies of C_μ , so by the same argument, each of these scycles has average weight ω , and thus is a leading scycle. \square

Let $\tilde{\mu}$ denote the least common multiple of the lengths of leading scycles of A ; i.e., every $\ell \in L(A)$ divides $\tilde{\mu}$, cf. (3.5). In particular, μ divides $\tilde{\mu}$.

Proposition 4.21. *Suppose C is an scycle in $\text{core}(A)$, with $i \in \mathcal{V}(C)$, of weight $w_C = w(C)$ and of length $\ell = \ell(C)$. Then, for any k , the (i, i) -diagonal entry of $A^{k\tilde{\mu}}$ is $(w_C)^{k\tilde{\mu}/\ell}$, which is ν -equivalent to $\omega^{k\tilde{\mu}}$. Furthermore, assuming that $\text{core}(A)$ is nonempty,*

$$|(A^{k\tilde{\mu}})_{\text{core}}| = \prod_{C \subset \text{core}(A)} (w_C)^{k\tilde{\mu}/\ell(C)},$$

i.e., $|(A^{k\tilde{\mu}})_{\text{core}}| \cong_\nu \omega^{ks\tilde{\mu}}$, where $s = \#(\mathcal{V}(\text{core}(A)))$.

Proof. There is only one dominant term in the (i, i) entry of $A^{k\tilde{\mu}}$, which comes from repeating the single ℓ -leading scycle $C(i, i)$ containing i (starting at position i) $k\tilde{\mu}/\ell$ times. Since $C(i, i)$ is a core-admissible leading scycle, $\sqrt[\ell]{C(i, i)} \cong_\nu \omega$. This proves the first assertion.

First assume for simplicity that $\text{core}(A)$ is comprised of a single scycle. Any other contribution to the determinant of $(A^{k\tilde{\mu}})_{\text{core}}$ would also come from a multicycle, and thus from a power of C , by assumption a unique leading cycle, which must then be the same product along the diagonal. Thus, the single leading multicycle of $A^{k\tilde{\mu}}$ is the one along the diagonal, which yields the determinant.

The same argument applies simultaneously to each core-admissible scycle. Namely, any dominant term along the diagonal must occur from repeating the same scycle, since the leading scycles are presumed disjoint, and again the single leading multicycle of $A^{k\tilde{\mu}}$ is the diagonal. \square

Corollary 4.22. *Assuming that $\text{tcore}(A)$ is nonempty, $|(A^{k\tilde{\mu}})_{\text{tcore}}| = \omega^{ks\tilde{\mu}}$, where $s = \#(\mathcal{V}(\text{tcore}(A)))$.*

Recall that the rank of a matrix is the maximal number of tropically independent rows (or columns), which is the same as the maximal size of a nonsingular submatrix, cf. [9].

Corollary 4.23. *The rank of every power of A is at least $\#(\mathcal{V}(\text{tcore}(A)))$. In particular, if $\text{tcore}(A)$ is nonempty, then A is not ghostpotent.*

Corollary 4.24. *Suppose that the leading scycles of A are disjoint. Then, for any k , and any vertex i of a leading ℓ -scycle C , the (i, i) -diagonal entry of $A^{k\ell}$ is $w(C)^k$, which is ν -equivalent to $\omega^{k\ell}$.*

Example 4.25. *Let*

$$A = \begin{pmatrix} 0 & 2 & 4 \\ 4 & 0 & -1 \\ 1 & 0 & 3^\nu \end{pmatrix}. \quad \text{Then } A^2 = \begin{pmatrix} 6 & 4 & 7^\nu \\ 4^\nu & 6 & 8 \\ 4^\nu & 3^\nu & 6^\nu \end{pmatrix} \quad \text{and } A^4 = \begin{pmatrix} 12 & 10^\nu & 13^\nu \\ 12^\nu & 12 & 14^\nu \\ 10^\nu & 9^\nu & 12^\nu \end{pmatrix}.$$

For the matrix A we have $\mu = 1$, $\alpha_\mu = 3^\nu$ and thus $\omega = 3$. Moreover, $\tilde{\mu} = 2$ and in this matrix $\text{core}(A) = \text{tcore}(A) = A$; thus $A^{2\tilde{\mu}} = A^4$ and hence

$$A^4 = \begin{pmatrix} 12 & 10^\nu & 13^\nu \\ 12^\nu & 12 & 14^\nu \\ 10^\nu & 9^\nu & 12^\nu \end{pmatrix} = 12 \begin{pmatrix} 0 & -2^\nu & 1^\nu \\ 0^\nu & 0 & 2^\nu \\ -2^\nu & -3^\nu & 0^\nu \end{pmatrix},$$

where the matrix on the right is idempotent.

4.3. Semi-idempotent matrices.

Definition 4.26. *A matrix $A \in M_n(R)$ is **semi-idempotent** if $A^2 = \beta A$ for some tangible $\beta = \beta(A)$ in R . We call $\beta(A)$ the **semi-idempotent coefficient** of A .*

Remark 4.27. *If A is a semi-idempotent matrix, then $A^k = A^{k-2}A^2 = A^{k-2}\beta A = \beta A^{k-1} = \dots = \beta^{k-1}A$, and any power of A is semi-idempotent.*

Lemma 4.28. *If A is semi-idempotent, then $\mu(A) = 1$.*

Proof. By Remark 4.27, $A^\mu = \beta^{\mu-1}A$, for any $\mu \geq 1$. Clearly, $\mu(\alpha A) = \mu(A)$ for any $a \in R$. Then, $\mu(A) = \mu(A^\mu) = 1$, by Lemma 4.14. \square

The next result ties this concept in with [10, 11].

Lemma 4.29. *If A is a nonsingular semi-idempotent matrix, then $|A| = \beta$ and $\beta^{-1}A$ is a quasi-identity matrix.*

Proof. Clearly $\beta^{-1}A$ is nonsingular semi-idempotent, since β^{-1} is tangible, and its determinant is tangible. So $|A^2| = |\beta A| = \beta|A|$ is tangible, implying

$$\beta|A| = |A^2| = |A|^2,$$

by Equation (3.1), and thus $\beta = |A|$. Hence, $|\beta^{-1}A| = \beta^{-1}|A| = \mathbb{1}_R$, implying the matrix $\beta^{-1}A$ is idempotent, and thus is a quasi-identity matrix by Proposition 3.3. \square

Example 4.30. *The matrix*

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

satisfies $A^2 = 4A$, so A is a singular semi-idempotent matrix.

Theorem 4.31.

- (i) *For any matrix A with nonempty core, the submatrix $(A^{m\tilde{\mu}})_{\text{core}}$ is semi-idempotent for some power m , with semi-idempotent coefficient $\beta((A^{m\tilde{\mu}})_{\text{core}}) = \omega^{m\tilde{\mu}}$.*
- (ii) *For any matrix A with nonempty tcore, the submatrix $(A^{m\tilde{\mu}})_{\text{tcore}}$ is semi-idempotent for some power m , with semi-idempotent coefficient $\beta((A^{m\tilde{\mu}})_{\text{tcore}}) = \omega^{m\tilde{\mu}}$, and hence $(\omega^{m\tilde{\mu}})^{-1}(A^{m\tilde{\mu}})_{\text{tcore}}$ is a quasi-identity matrix.*

Proof. (i) The (i, j) entry $b_{i,j}$ of $B = (A^{m\tilde{\mu}})_{\text{core}}$ is obtained from some path p of length $m\tilde{\mu}$ from i to j in the digraph G_A , from which we extract as many scycles as possible; cf. Remark 4.17, to arrive at some simple path p' from i to j without scycles; thus p' has length $\leq n$, and length $< n$ when $i \neq j$. p , as well as each of whose scycles, has a ghost weight iff it has a ghost edge. Furthermore, by Lemma 4.20, given a leading ℓ -scycle C in G_A , we could replace $C^{\tilde{\mu}}$ by $\omega^{\ell\tilde{\mu}}$ or $(\omega^{\ell\tilde{\mu}})^\nu$, depending whether C is ghost or not, without decreasing the ν -value of $b_{i,j}$; hence we may assume that it is possible to extract at most $\tilde{\mu} - 1$ scycles of length $\ell \neq \mu$ from p , for each ℓ .

Working backwards, we write $p'_\ell(i, j)$ for the path of length ℓ from i to j in G_A having maximal weight. There are only finitely many maximal possibilities for multiplying this by weights of scycles of length $\neq \mu$, so at some stage, taking higher powers m only entails multiplying by ω^μ . Doing this for each pair (i, j) yields the theorem.

(ii) The same argument as in (i), noting that the diagonal now is tangible. \square

In case A is an irreducible matrix, it is known that $A^{m\tilde{\mu}+1} = \omega^{m\tilde{\mu}}A$ over the max-plus algebra; cf. [1, §25.4, Fact 2(b)], where m is called the **cyclicity**. This does not quite hold in the supertropical theory, because of the difficulty that taking powers of a matrix might change tangible terms to ghost. Thus, we must settle for the following result.

Corollary 4.32. *In case A is irreducible, $A^{m\tilde{\mu}+1} \cong_\nu \omega^{m\tilde{\mu}}A$ and $A^{km\tilde{\mu}+1} = \omega^{(k-1)m\tilde{\mu}}A^{m\tilde{\mu}+1}$ for all $k > 1$.*

Proof. Take m as in Theorem 4.31(i). Let $c_{i,j}$ denote the (i, j) entry of $A^{m\tilde{\mu}+1}$. To prove the first assertion, we need to show that $c_{i,j} = \omega^{m\tilde{\mu}}a_{i,j}$. We take a maximal path p from i to j in the graph of $A^{m\tilde{\mu}+1}$. Let $b_{i,j}$ denote the (i, j) entry of $A^{m\tilde{\mu}}$, the weight of the path p . Then $c_{i,j} \leq_\nu \omega^{m\tilde{\mu}}a_{i,j}$ by Proposition 4.21, since every cycle of length μ must have weight $\leq \omega^{m\tilde{\mu}}$. On the other hand, Lemma 4.20 gives us leading scycles of total length $\ell\mu$. This yields $\omega^{m\tilde{\mu}}a_{i,j} \leq \omega^{\ell\mu}b_{i,j} \leq_\nu c_{i,j}$; indeed any cycle of length μ in p must have weight $\omega^{m\tilde{\mu}}$ (since otherwise it could be replaced by C , thereby providing a path of weight $> w(p)$, a contradiction). Thus, equality holds and we proved the first assertion.

The second assertion follows, by taking one more pass through the scycles, as illustrated in Example 4.5. \square

We obtain a generalization in the non-irreducible case, in Theorem 5.7.

Corollary 4.33. *Any matrix in full block triangular form has a power such that each diagonal block is semi-idempotent.*

Proof. Apply Theorem 4.31(ii) to each diagonal block. \square

Corollary 4.34. *$(A^{m\tilde{\mu}})_{\text{core}}$ is $\omega^{m\tilde{\mu}}$ times an idempotent matrix $J_{\mathcal{G}} \models_{\text{gs}} I_{\mathcal{G}}$, where \mathcal{G} is a quasi-identity matrix, for some m .*

Proof. Dividing out by $\omega^{m\tilde{\mu}}$, we take a suitable power and may assume that $(A)_{\text{core}}$ is idempotent. But we can also replace each of the diagonal entries of $(A)_{\text{core}}$ by $\mathbb{1}_R$ or $\mathbb{1}_R^\nu$, and then are done by Corollary 4.32. \square

Corollary 4.35. *$(A^{m\tilde{\mu}})_{\text{tcore}}$ is $\omega^{m\tilde{\mu}}$ times a quasi-identity matrix, for some m .*

Proof. Consequence of Corollary 4.34. \square

Example 4.36. *The matrix*

$$A = \begin{pmatrix} \mathbb{0}_R & \mathbb{1}_R \\ \mathbb{1}_R & \mathbb{0}_R \end{pmatrix}$$

satisfies $A^{2n+1} = A$ but $A^{2n-1}A = A^{2n} = I$ for each n .

Note that in Example 4.3, A^2 is semi-idempotent and not ghostpotent, but singular. Let us now consider scycles in the anti-tcore.

Proposition 4.37. *Suppose $\rho_i > 1$. Then, for any $k \geq 2$, the (i, i) -diagonal entry of $(A^{k\bar{\mu}})_{\text{anti-tcore}}$ is $(\omega^{k\bar{\mu}})^\nu$. Furthermore, when $\text{anti-tcore}(A)$ is nonempty,*

$$|(A^{k\bar{\mu}})_{\text{anti-tcore}}| = (\omega^{sk\bar{\mu}})^\nu,$$

where $s = \#(\mathcal{V}(\text{anti-tcore}(A)))$.

Proof. There are at least two dominant terms in the (i, i) entry of $(A^{k\bar{\mu}})_{\text{anti-tcore}}$, which come from exchanging two leading scycles $C(i, i)$ containing i (starting at position i), or a dominant ghost term which comes from a leading ghost scycle containing i . \square

Theorem 4.38. *If $\text{anti-tcore}(A)$ is nonempty, then some power of A is singular. If A is irreducible and $\text{tcore}(A)$ is empty, then A is ghostpotent.*

More generally, for A irreducible, there is a power of A such that the (i, j) entry of A is ghost unless $i, j \in \mathcal{V}(\text{tcore}(A))$.

Proof. The diagonal elements from $(A^{m\bar{\mu}})_{\text{tcore}}$ and $(A^{m\bar{\mu}})_{\text{anti-tcore}}$ occur in the multicycle determining $|A^{m\bar{\mu}}|$ for large m , yielding the first assertion. To prove the last assertion (which implies the second assertion), we need to show that every (i, j) entry of $A^{m\bar{\mu}}$ involves a leading scycle, for m sufficiently large, but this is clear from Remark 4.17, since A is irreducible. \square

Theorem 4.39. *A matrix A is ghostpotent iff the submatrix of each of its strongly connected components is ghostpotent according to the criterion of Theorem 4.38, in which case the index of ghostpotence is at most the number of strongly connected components times the maximal index of ghostpotence of the strongly connected components.*

Proof. We write A in full block triangular form, and then apply Lemma 4.8. \square

5. THE JORDAN DECOMPOSITION

We are ready to find a particularly nice form for powers of A .

Definition 5.1. *A matrix A is in **stable block triangular form** if*

$$A = \begin{pmatrix} B_1 & B_{1,2} & \dots & B_{1,\eta-1} & B_{1,\eta} \\ (0) & B_2 & \dots & B_{2,\eta-1} & B_{2,\eta} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (0) & \dots & (0) & B_{\eta-1} & B_{\eta-1,\eta} \\ (0) & \dots & (0) & (0) & B_\eta \end{pmatrix}$$

is in full block triangular form, such that each B_i is semi-idempotent and

$$A^2 = \begin{pmatrix} \beta_1 B_1 & \beta_{1,2} B_{1,2} & \dots & \beta_{1,\eta-1} B_{1,\eta-1} & \beta_{1,\eta} B_{1,\eta} \\ (0) & \beta_2 B_2 & \dots & \beta_{2,\eta-1} B_{2,\eta-1} & \beta_{2,\eta} B_{2,\eta} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (0) & \dots & (0) & \beta_{\eta-1} B_{\eta-1} & \beta_{\eta-1,\eta} B_{\eta-1,\eta} \\ (0) & \dots & (0) & (0) & \beta_\eta B_\eta \end{pmatrix},$$

where $\beta_i = \beta(B_i)$ and $\beta_{i,j} \in \{\beta_i, \beta_i^\nu, \dots, \beta_j, \beta_j^\nu\}$ for each $i < j$. If each $\beta_{i,j} \in \mathcal{T}$, we say that A is in **tangibly stable block triangular form**.

Definition 5.2. *A matrix S is **semisimple** if $S^{2k} = DS^k$ for some tangible diagonal matrix D and $k \in \mathbb{N}$. We say that A has a **Jordan decomposition** if $A = S + N$ where S is semisimple and N is ghostpotent.*

Obviously, any semi-idempotent matrix is semisimple.

Lemma 5.3. *Suppose*

$$A = \begin{pmatrix} B_1 & B_{1,2} & \cdots & B_{1,\eta-1} & B_{1,\eta} \\ (0) & B_2 & \cdots & B_{2,\eta-1} & B_{2,\eta} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (0) & \cdots & (0) & B_{\eta-1} & B_{\eta-1,\eta} \\ (0) & \cdots & (0) & (0) & B_\eta \end{pmatrix}$$

is in full block triangular form. If $B_i = S_i + N_i$ is a Jordan decomposition for B_i for $i = 1, \dots, \eta$, then

$$A = \begin{pmatrix} S_1 & (0) & \cdots & (0) & (0) \\ (0) & S_2 & \cdots & (0) & (0) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (0) & \cdots & (0) & S_{\eta-1} & (0) \\ (0) & \cdots & (0) & (0) & S_\eta \end{pmatrix} + \begin{pmatrix} N_1 & B_{1,2} & \cdots & B_{1,\eta-1} & B_{1,\eta} \\ (0) & N_2 & \cdots & B_{2,\eta-1} & B_{2,\eta} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (0) & \cdots & (0) & N_{\eta-1} & B_{\eta-1,\eta} \\ (0) & \cdots & (0) & (0) & N_\eta \end{pmatrix}$$

is a Jordan decomposition for A .

Proof.

Clearly $\begin{pmatrix} S_1 & (0) & \cdots & (0) & (0) \\ (0) & S_2 & \cdots & (0) & (0) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (0) & \cdots & (0) & S_{\eta-1} & (0) \\ (0) & \cdots & (0) & (0) & S_\eta \end{pmatrix}$ is semisimple, and $\begin{pmatrix} N_1 & B_{1,2} & \cdots & B_{1,\eta-1} & B_{1,\eta} \\ (0) & N_2 & \cdots & B_{2,\eta-1} & B_{2,\eta} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (0) & \cdots & (0) & N_{\eta-1} & B_{\eta-1,\eta} \\ (0) & \cdots & (0) & (0) & N_\eta \end{pmatrix}$ is ghost-

potent, by Lemma 4.8. \square

Remark 5.4. *For any matrix $A \in M_n(R)$ in full block triangular form, we view A as acting on $R^{(n)}$ with respect to the standard basis e_1, \dots, e_n . The diagonal block B_j uses the columns and rows say from i_j to $i_{j+1} - 1$, and acts naturally on the subspace V_j generated by $e_{i_j}, \dots, e_{i_{j+1}-1}$. Thus, we have the natural decomposition $R^{(n)} = V_1 \oplus \cdots \oplus V_\eta$. We view each V_j as a subspace of $R^{(n)}$ under the usual embedding, and the i_j to $i_{j+1} - 1$ columns of A act naturally on V_j .*

Theorem 5.5. *For any matrix A in full block triangular form of length η for which the diagonal blocks are semi-idempotent, the matrix A^η has stable block triangular form. Furthermore, $\beta_{i,j} \in \{\alpha, \alpha^\nu\}$, where*

$$\beta = \sum (\beta_{i_1} + \beta_{i_2} + \cdots + \beta_{i_k}),$$

summed over all paths (i_1, \dots, i_k) such that $i_1 = i$ and $i_k = j$. (Thus $k \leq j - i$). In other words, $\beta_{i,j}$ is the maximum semi-idempotent coefficient that appears in a path from i to j .

More generally, under the given decomposition $R^{(n)} = V_1 \oplus \cdots \oplus V_\eta$ such that, for any vector v , writing $A^\eta v = (v_1, \dots, v_\eta)$ for $v_i \in V_i$, one has

$$A(A^\eta v_j) = \sum_{i=1}^j \bar{\beta}_{i,j} v_i, \quad (5.1)$$

where $\bar{\beta}_{i,j}$ is the maximum of the β_{i_k} (or its ghost).

Proof. Write

$$A = \begin{pmatrix} B_1 & B_{1,2} & \cdots & B_{1,\eta-1} & B_{1,\eta} \\ (0) & B_2 & \cdots & B_{2,\eta-1} & B_{2,\eta} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (0) & \cdots & (0) & B_{\eta-1} & B_{\eta-1,\eta} \\ (0) & \cdots & (0) & (0) & B_\eta \end{pmatrix} \quad \text{and} \quad A^\eta = \begin{pmatrix} \tilde{B}_1 & \tilde{B}_{1,2} & \cdots & \tilde{B}_{1,\eta-1} & \tilde{B}_{1,\eta} \\ (0) & \tilde{B}_2 & \cdots & \tilde{B}_{2,\eta-1} & \tilde{B}_{2,\eta} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (0) & \cdots & (0) & \tilde{B}_{\eta-1} & \tilde{B}_{\eta-1,\eta} \\ (0) & \cdots & (0) & (0) & \tilde{B}_\eta \end{pmatrix}.$$

Then $\tilde{B}_j = B_j^\eta$ for each $j = 1, \dots, \eta$, and for $i < j$,

$$\tilde{B}_{i,j} = \sum B_{i_1}^{u_1} B_{i_1, i_2} B_{i_2}^{u_2} \cdots B_{i_{\ell-1}, i_\ell} B_{i_\ell}^{u_\ell}, \quad (5.2)$$

summed over all i_1, \dots, i_ℓ where $i_1 = i$ and $i_\ell = j$. (Here $\ell \leq j - i$.) We take a typical summand $B_{i_1}^{u_1} B_{i_1, i_2} B_{i_2}^{u_2} \cdots B_{i_{\ell-1}, i_\ell} B_{i_\ell}^{u_\ell}$. By assumption, when $u_k > 1$ we may replace $B_{i_k}^{u_k}$ by $\beta_{i_k}^{u_k-1} B_{i_k}$. But then we could replace $B_{i_k}^{u_k}$ by $B_{i_{k'}}^{u_k}$ for any k' . So, taking $\beta = \sum_{k=1}^\ell \beta_{i_k}$, which is ν -equivalent to some $\beta_{i_{k'}}$, we also have the term

$$\beta^{\eta-\ell} B_{i_1, i_2} \cdots B_{i_{\ell-1}, i_\ell}$$

and thus we may assume that $u_k = 0$ (in (5.2)) for each $k \neq k'$. In other words, we sum over all paths $p := B_{i_1, i_2}, \dots, B_{i_{\ell-1}, i_\ell}$, where $i_1 = i$, $i_\ell = j$, and the coefficient $\beta^{\eta-\ell}$ comes from $B_{i_k}^{\eta-\ell}$, where i_k appears in $\mathcal{V}(p)$.

Note that $\ell < \eta$, so if there are two possibilities for k' we get two equal maximal paths and thus get a ghost value for $\beta_{i,j}$ (and likewise if there is one maximal ghost path, or if different maximal paths are ν -equivalent). If there is one single path p of maximal weight, which is tangible, and if p is tangible, then $\beta_{i,j}$ is tangible.

The same argument yields the last assertion when we consider $A^\eta v_j$. \square

Corollary 5.6. *Hypotheses as in Theorem 5.5 $A^{2\eta}$ is in tangibly stable block triangular form.*

Proof. All the ghost entries already occur in $A^{2\eta}$, so we can replace any ghost $\beta_{i,j}$ by $\widehat{\beta_{i,j}}$. \square

We now are ready for one of the major results.

Theorem 5.7. *For any matrix A , there is some power m such that A^m is in tangibly stable block triangular form.*

Proof. A can be put into full block triangular form by Proposition 3.5, and a further power is in tangibly stable block triangular form, by Theorem 5.7 and Corollary 4.33. \square

We call this m the **stability index** of A .

Example 5.8. *There is no bound (with respect to the matrix's size) on the stability index of A . Indeed, in Example 4.9 we saw a 2×2 matrix A such that A^{m-1} is nonsingular but A^m is singular, where m can be arbitrarily large.*

Theorem 5.9. *Any matrix $A \in M_n(R)$ has a Jordan decomposition, where furthermore, in the notation of Definition 5.2, $|A| = |S|$.*

Proof. In view of Lemma 5.3, it suffices to assume that A is irreducible. Then, we conclude with Corollary 4.32. \square

Example 5.10. *The matrix (in logarithmic notation)*

$$A = \begin{pmatrix} 10 & 10 & 9 & - \\ 9 & 1 & - & - \\ - & - & - & 9 \\ 9 & - & - & - \end{pmatrix}$$

of [11, Example 5.7] (the empty places stand for $-\infty$) is semisimple, but the tangible matrix B given there must be taken to be nonsingular.

6. SUPERTROPICAL GENERALIZED EIGENVECTORS AND THEIR EIGENVALUES

We started studying supertropical eigenspaces in [10], and saw how to calculate supertropical eigenvectors in [11], but also saw that the theory is limited even when the characteristic polynomial factors into tangible linear factors. To continue, we need to consider generalized supertropical eigenvectors. We recall [10, Definition 7.3].

Definition 6.1. *A tangible vector v is a **generalized supertropical eigenvector** of A , with **generalized supertropical eigenvalue** $\beta \in \mathcal{T}_0$, if*

$$A^m v \models_{\text{gs}} \beta^m v$$

for some m ; the minimal such m is called the **multiplicity**. A **supertropical eigenvalue** (resp. **supertropical eigenvector**) is a generalized supertropical eigenvalue (resp. generalized supertropical eigenvector) of multiplicity 1. A vector v is a **strict eigenvector** of A , with **eigenvalue** $\beta \in \mathcal{T}_0$, if $Av = \beta v$.

Recall, cf. [7, Definition 3.1], that a vector $v \in R^{(n)}$ is a **g-annihilator** of A if $Av \in \mathcal{G}_0^{(n)}$, i.e., $Av \models \bar{0}$. A **tangible g-annihilator** is a g-annihilator that belongs to $\mathcal{T}_0^{(n)}$. (Accordingly, any tangible g-annihilator of A is the same as a supertropical eigenvector with supertropical eigenvalue 0_R .) The **ghost kernel** of A is defined as

$$\text{g-ker}(A) := \{v \in R^{(n)} \mid Av \in \mathcal{G}_0^{(n)}\};$$

in particular $\mathcal{G}_0^{(n)} \subset \text{g-ker}(A)$ for any A . If A is a ghost matrix, then $\text{g-ker}(A) = R^{(n)}$.

Example 6.2. Any quasi-identity matrix $A = I_{\mathcal{G}}$ has n tropically independent strict eigenvectors, each with eigenvalue 1_R , namely the columns of A (since A is idempotent and nonsingular). Likewise, any nonsingular semi-idempotent matrix has n tropically independent strict eigenvectors, each with eigenvalue $\beta(A)$.

When A is not necessarily nonsingular, we still have an analogous result.

Proposition 6.3. For any irreducible, semi-idempotent $n \times n$ matrix A , if $s = \#(\mathcal{V}(\text{tcore}(A)))$, the s columns of the submatrix $(A)_{\text{tcore}}$ (corresponding to $\text{tcore}(A)$) are tropically independent, strict eigenvectors of $(A)_{\text{tcore}}$, and are also supertropical eigenvectors of A , which can be expanded to a set of n tropically independent vectors of $R^{(n)}$, containing $n - s$ tangible g-annihilators of A .

Proof. Replacing A by $\beta^{-1}A$, where $\beta = \beta(A)$, we may assume that A is an idempotent matrix. Let U denote the subspace of $R^{(n)}$ corresponding to $(A)_{\text{tcore}}$. If v is a column of A , and $v' = v|_U$ is its restriction to a column of U , then clearly

$$(A)_{\text{tcore}}v' \leq (Av)|_U = v' = Iv' \leq (A)_{\text{tcore}}v',$$

implying $(A)_{\text{tcore}}v' = (Av)|_U = v'$.

These vectors v are also supertropical eigenvectors of A , since the other components of Av are ghost, in view of Theorem 4.38.

To prove the last assertion, we repeat the trick of [7, Proposition 4.12]). Rearranging the base, we may assume that $\mathcal{V}(\text{tcore}(A)) = \{1, \dots, s\}$. For any other row v_u of A ($m < u \leq n$), we have $\beta_{u,1}, \dots, \beta_{u,m} \in \mathcal{T}_0$ such that $v_u + \sum \beta_{i,j}v_i \in \mathcal{G}_0^{(n)}$.

Let B' be the $(n - m) \times n$ matrix whose first s columns are the s columns of $(A)_{\text{tcore}}$ (with (i, j) -entry 0_R for $i > s$) and whose entries (i, j) are $\beta_{i,j}$ for $1 \leq i, j \leq m$, and for which $\beta_{i,j} = \delta_{i,j}$ (the Kronecker delta) for $m < j \leq n$. Then B' is block triangular with two diagonal blocks, one of which is the identity matrix, implying $|B'| = |(A)_{\text{tcore}}|$ and thus B' is nonsingular. This gives us the desired n tropically independent supertropical eigenvectors. \square

Lemma 6.4. If $A^m v \underset{\text{gd}}{\gamma} \beta^m v$ for a tangible vector v , some m , and $\beta \in \mathcal{T}_0$, then v is a generalized supertropical eigenvector of A of multiplicity m , with generalized supertropical eigenvalue β .

Proof. The vector $\beta^m v$ is tangible, so clearly $A^m v \underset{\text{gs}}{\models} \beta^m v$ (cf. [7, Lemma 2.9]). \square

Lemma 6.5. If β is a generalized supertropical eigenvalue for A of multiplicity m , then β also is a generalized supertropical eigenvalue for A of multiplicity m' , for each multiple m' of m .

Proof.

$$A^{km} v = A^{(k-1)m} A^m v \underset{\text{gs}}{\models} A^{(k-1)m} \beta^m v = \beta^m A^{(k-1)m} v \underset{\text{gs}}{\models} \beta^{km} v,$$

by induction. \square

Proposition 6.6. The generalized supertropical eigenvectors corresponding to a supertropical eigenvalue β form a subspace $V_\beta(A) \subset R^{(n)}$ which is A -invariant.

Proof. If $v, w \in V_\beta(A)$, then

$$A^m v \models_{\text{gs}} \beta^m v, \quad A^{m'} w \models_{\text{gs}} \beta^{m'} w,$$

for suitable m, m' , so taking their maximum m'' yields $A^{m''}(v+w) \models_{\text{gs}} \beta^{m''}(v+w)$, and likewise for scalar products, implying $\alpha v \in V_\beta(A)$, for any $\alpha \in R$.

Also,

$$A^m(Av) = A(A^m v) \models_{\text{gs}} A(\beta^m v) = \beta^m(Av),$$

and thus $Av \in V_\beta(A)$. \square

We call this space $V_\beta(A)$ the **generalized supertropical eigenspace** of β . This is easiest to describe when A is nonsingular.

Theorem 6.7. *Suppose a nonsingular matrix A is in stable block triangular form, notation as in Definition 5.1, and write $V = V_1 \oplus \cdots \oplus V_\eta$ where each V_i has rank n_i and*

$$Av_j = \sum_i B_{i,j} v_i, \quad \forall v_j \in V_j.$$

*Then there are supertropical eigenspaces \tilde{V}_j of A with respect to supertropical eigenvalues β_j , such that $V_A := \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_\eta$ is a **thick** subspace of V in the sense of [7, Definition 5.28] (which means that V_A also has rank n).*

Proof. Each diagonal block B_j is nonsingular. Let V'_j denote the subspace of V_j spanned by the rows of B_j . In other words,

$$V'_j := \{B_j v : v \in V\},$$

a thick subspace of V in view of [7, Remark 6.14], since B_j behaves like a quasi-identity matrix in view of Lemma 4.29.

Now for each $v \in V'_j$ we write $Av = \sum_{i=1}^\eta v_i$ where $v_i \in V_i$. By Theorem 5.5,

$$Av_j = \sum_{i=1}^j \bar{\beta}_{i,j} v_i$$

for $v_j \in \hat{V}_j$. Starting with $i = j$ we put $\tilde{v}_{j,j} = v_j$ and, proceeding by reverse induction, given $\tilde{v}_{k,j}$ for $i < k \leq j$ take

$$\tilde{v}_{i,j} = \sum_{k=i+1}^j \frac{\bar{\beta}_{k,j}}{\beta_i} \tilde{v}_{k,j}.$$

We put

$$\tilde{v}_j = \tilde{v}_{1,j} + \cdots + \tilde{v}_{j,j}.$$

Then for each $i < j$ the i -component of $A\tilde{v}_j$ is

$$\sum_{k=i+1}^j \left(\bar{\beta}_{k,j} \tilde{v}_{k,j} + \beta_i \frac{\bar{\beta}_{k,j}}{\beta_i} \tilde{v}_{k,j} \right) = \sum_{k=i+1}^j \bar{\beta}_{k,j}^\nu \tilde{v}_{k,j},$$

whereas the j -component of $A\tilde{v}_j$ is $\beta_j \tilde{v}_j$. Hence, $A\tilde{v}_j \models_{\text{gs}} \beta_j \tilde{v}_j$, as desired. \square

When A need not be nonsingular, we need to modify the assertion slightly.

Theorem 6.8. *Suppose the matrix A is in stable block triangular form, notation as in Definition 5.1, and write $V = V_1 \oplus \cdots \oplus V_\eta$ where each V_i has rank n_i and*

$$Av_j = \sum_i B_{i,j} v_i, \quad \forall v_j \in V_j.$$

Let $s_j = \#(\mathcal{V}(\text{tcore}(B_j)))$. Then there are supertropical eigenspaces \tilde{V}_j of A with respect to supertropical eigenvalues β_j , as well as a g -annihilator space V_0 , such that $V_A := \tilde{V}_0 \oplus \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_\eta$ is a thick subspace of V .

Proof. We repeat the proof of Theorem 6.7 noting that when B_j is singular, one could take \widehat{B}_j to be the space of Proposition 6.3, which provides extra g -annihilating vectors in each component, but does not affect the rest of the argument. \square

6.1. Weak generalized supertropical eigenspaces. Generalized eigenspaces are understood better when we introduce the following related notion. (We write $\bar{0}$ for the zero vector in $R^{(n)}$.)

Definition 6.9. A vector $v \neq \bar{0}$ is a **weak generalized supertropical eigenvector** of A , with (tangible) **weak m -generalized supertropical eigenvalue** $\beta \in \mathcal{T}_0$, if

$$(A^m + \beta^m I)^k v \underset{\text{gs}}{=} \bar{0}$$

for some k .

Remark 6.10. Any generalized supertropical eigenvector is a weak m -generalized supertropical eigenvector, in the view of Remark 2.1.

Lemma 6.11. If β is a weak m -generalized supertropical eigenvalue for A , then β also is a weak m' -generalized supertropical eigenvalue for A , for each m' dividing m .

Proof. Write $m = m'd$.

$$(A^{m'} + \beta^{m'} I)^d \underset{\text{gs}}{=} A^{m'd} + \beta^{m'd} I = A^m + \beta^m I,$$

by Proposition 2.1, yielding $(A^{m'} + \beta^{m'} I)^{dk} v \underset{\text{gs}}{=} (A^m + \beta^m I)^k v$. Thus, v is a weak m' -generalized supertropical eigenvector for A . \square

Just as with Proposition 6.6, we have (with the analogous proof):

Proposition 6.12. The weak m -generalized supertropical eigenvectors corresponding to a root β form an A -invariant subspace of $R^{(n)}$.

We call this space of Proposition 6.12 the **weak m -generalized eigenspace** of β . Considering A as a linear operator acting on $R^{(n)}$, the weak m -generalized eigenspace is the union of the ascending chain of subspaces

$$g\text{-ker}(A^m + \beta^m I) \subseteq g\text{-ker}(A^m + \beta^m I)^2 \subseteq \dots$$

The following technique gives us a method to compute weak generalized eigenvectors.

Remark 6.13. Suppose A^m satisfies a polynomial $f = \prod f_i$, where each f_i is monic a_i -primary with constant term $\beta_i^{n_i}$, and for each $1 \leq j \leq t$ let

$$g_j = \prod_{i \neq j} f_i = \frac{f}{f_j}.$$

Then for each $v \in g_j(A)R^{(n)}$,

$$(A^m + \beta_j I)^{n_j} v \underset{\text{gs}}{=} f(A^m) v \underset{\text{gs}}{=} 0_F,$$

implying v is a weak m -generalized eigenvector of A , with eigenvalue β_j .

This gives us a weak m -generalized eigenspace of A clearly containing the generalized eigenspace $V_{\beta_j}(A)$, and leads us to explore the connection between these two notions.

Lemma 6.14. Suppose v is a weak m -generalized supertropical eigenvector of an irreducible matrix A of stability index m' , and supertropical eigenvalue β . Suppose $q = dm'$, and suppose $A^{2m'} = \gamma A$. Let

$$v' = \sum_{j=0}^{m'-1} A^j \beta^{(m'-j)} v.$$

Then

$$\sum_{j=0}^q (A + \beta I)^j v = \begin{cases} \beta^d v' & \text{for } \beta >_\nu \gamma, \\ (\beta^d)^\nu v' & \text{for } \beta \cong_\nu \gamma, \\ \gamma^d v' & \text{for } \beta <_\nu \gamma. \end{cases}$$

Proof. This is immediate from Proposition 2.2. \square

We are ready to show that the behavior of weak generalized supertropical eigenvectors is “controlled” by the stability index of A .

Theorem 6.15. *Given a matrix A with stability index m' , suppose $v \in V_A$ is a weak generalized supertropical eigenvector of A with weak generalized supertropical eigenvalue β . Then β is a generalized supertropical eigenvalue of $A^{m'}$, and $\sum_{j=0}^{2m'} (A + \beta I)^j v$ is already a ghost vector.*

Proof. Decomposing v as in Theorem 5.5, we may assume that v is some V_j . Take v' as in Lemma 6.14. If $\beta \neq \beta_j$, then we get a tangible component in $(A + \beta I)^q v$ for high enough powers of q , contrary to assumption. Hence $\beta = \beta_j$, and again we conclude with Lemma 6.14. \square

So we see that the “difference” between weak generalized supertropical eigenvalues and generalized supertropical eigenvalues occurs within twice the stability index. (We could lower this bound with some care.)

Example 6.16.

Here is an example which illustrates some new pitfalls. We take $A = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$ as in Example 4.3. Clearly $0 = 0^2$ and $4 = 2^2$ are supertropical eigenvalues of A^2 , but now, in view of [10, Proposition 7.7], every tangible $\beta \leq_\nu 1$ is a supertropical eigenvalue of

$$A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},$$

since β is a root of $f_{A^2} = \lambda^2 + 4\lambda + 5^\nu$. Let us compute the tangible eigenvectors, using the methods of [7].

The singular matrix A^2 has adjoint $\begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}$ and thus the g -annihilator $v = (2, 1)^t$, which can be checked by noting that $A^2 v = (3^\nu, 5^\nu)^t$, which is ghost. From this point of view, $(2, 1)^t$ is a generalized supertropical eigenvector for A having eigenvalue $-\infty$ of multiplicity 2, although it is also a g -annihilator of A^2 .

Note that $A^2 + \beta I = A^2$ for all $\beta <_\nu 1$. From this point of view, these β are “phony” generalized eigenvalues of A .

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